

**General Covariance and the “Problem of Time” in a Discrete Cosmology\****Graham Brightwell**Mathematics Department, London School of Economics**H. Fay Dowker**Physics Department, Queen Mary College, University of London**Raquel S. García**Physics Department, Imperial College, London**Joe Henson**Physics Department, Queen Mary College, University of London**Rafael D. Sorkin**Physics Department, Queen Mary College, University of London  
and**Department of Physics, Syracuse University***Abstract**

Identifying an appropriate set of “observables” is a nontrivial task for most approaches to quantum gravity. We describe how it may be accomplished in the context of a recently proposed family of stochastic (but classical) dynamical laws for causal sets. The underlying idea should work equally well in the quantum case.

**1. Introduction**

Perhaps I should begin by clarifying what I *don’t* mean by the quoted phrase “problem of time” in my title. In the canonical quantization of gravity, as it is normally understood (and sought) the fundamental object of attention is not spacetime but space alone

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(corresponding to something like a Cauchy surface in spacetime) and anyone following this approach is sooner or later faced with the problem of *recovering time* from a frozen formalism (as it's sometimes called) from which time as such is absent. The problem of time *in this sense* is, I believe, insoluble and it will not be the subject of this talk. (See [1] [2].) Indeed, since I will be presupposing that the deep structure of spacetime is that of a *causal set*, temporality will be built in at the most fundamental level, and there will be no need to “recover” it, just as there would be no need to “recover time” in a continuum path integral approach to quantum gravity based on the Lorentzian manifold as the fundamental structure. \*

Nevertheless, there still remain vital interpretational issues related to the generally covariant nature of gravity that sometimes are also called “problems of time”,<sup>†</sup> issues that one can point to by asking (in the language of one version of quantum mechanics) “What are the observables of quantum gravity?”. For my part, I'd prefer a reference to “be-ables” rather than “observe-ables”, but the question I will be addressing in this talk is best posed without using either word. Rather we can simply ask “To what questions (about the causal set) can the dynamics give answers?”

Within the (stochastic but still classical) dynamical framework we will be considering, a *question* (in logicians' language a “predicate”) corresponds to a collection of causal sets (the predicate's “extension”) and its answer is not a simple yes or no, but the *probability* that the answer will be yes, which mathematically is the *measure* of the corresponding collection. Thus, we will be asking which classes of causal sets (the “histories” of the theory) are *measurable* in a way compatible with general covariance. I will propose a very definite answer to this question that will follow naturally from the way the dynamics will be defined as a stochastic process. Not all the resulting measurable classes (or their associated predicates) will obviously possess an accessible physical meaning, but a certain

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\* Not that, in saying this, I mean to downplay the need to explain why the dynamics tends to favor that small minority of causal sets which resemble Lorentzian manifolds over the multitude of those which don't.

<sup>†</sup> Observe in this connection that it is the diffeomorphism invariance of the gravitational Lagrangian (or rather action-integral) which is responsible for the “frozen” character of the corresponding canonical formalism

subset will (those based on “stems”). Fortunately this subset is very big, and it seems quite possible that nothing of interest — or even nothing at all — remains outside of it. After describing a precise conjecture to this effect, I will conclude by asking whether a proof of the conjecture would settle all the interpretational issues, or whether a logically independent notion of “conditional probability” is also needed. Before I can present these thoughts more fully, however, some review is needed of the causal set idea itself.

## 2. A Brief review of causal set theory

The causal set hypothesis states that the deep structure of spacetime is that of a discrete partial order, and that, consequently, “quantum gravity” can be realized only as a quantum theory of causal sets (“causets” for short). To say what a causet is structurally is easy: it’s a locally finite partial order; but to specify fully the meaning of the words “quantum theory of causets” is much harder. It seems plausible that the dynamics of such a theory (its “laws of motion” if you will) would be specified mathematically in terms of a “decoherence functional” [3] or “quantum measure” [4] but we are only beginning to understand the principles that might lead us to the correct one.<sup>b</sup> On the other hand, we do have a family of *classical* dynamical laws derived from well defined general principles including a principle of “discrete general covariance” and a certain principle of “Bell causality”. To the extent that these principles carry over to the quantal case, we are thus a considerable way along the road to a quantum dynamics for causets. Moreover, the projected quantum dynamics shares enough attributes with the existing classical one that it seems worthwhile to consult the latter for indications of what we can expect from its quantum generalization.

In this way, it has been possible to make some guesses and heuristic predictions which have begun to bring the theory into contact with phenomenology. The list of these must include first of all the anticipation [5] [2] [6] of time-dependent fluctuations in the cosmological constant whose predicted current magnitude of  $10^{-120}$  in natural units has turned

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<sup>b</sup> As I am using it, the term “decoherence functional” denotes no more than a certain type of mathematical object, formally defined by axioms of bi-additivity, etc. In particular, I do not mean to add any requirement that it actually “decohere” in the sense of being diagonal on any partition of the “sample space”  $\Omega$  to be defined below.

out to be in good accord with recent observations. Second [7] there exists a (purely kinematical) counting of “horizon quanta” whose number is (both in order of magnitude and proportionality to horizon area) compatible with the Bekenstein-Hawking formula for both equilibrium *and* nonequilibrium examples of black holes. We also have an indication [8] of how some of the notorious large numbers of cosmology might be explained, as well as a framework [9] within which Hawking radiation can be addressed. (In the way of practical tools there also exists an extensive library [10] of Lisp functions designed for working with causets, i.e. basically with finite ordered sets or “posets”.)

In the following, however, I wish to concentrate not on the phenomenological aspects of the theory but on interpretational ones, more specifically on the question already raised of which physically meaningful (i.e. generally covariant) predicates correspond to classes of causets to which the dynamics can assign a measure (probability). To this end, I will have to describe more precisely the family of stochastic dynamical laws on which these considerations will be based.

### **3. The classical (stochastic) dynamics of sequential growth and its formal definition as a stochastic process**

As indicated by the title of this talk, we will work within a “cosmological” setting, in the sense that the probabilities in question will pertain to the causet as a whole, not just to some part of it. This seems to be necessary, as it is difficult to imagine how any generally covariant procedure could single out a definite “subregion” of the universe in an *a priori* manner. A cosmological standpoint is also appropriate formally, since the dynamical laws we will be using conceive of the “time-development” of the causet as a process of sequential growth in which elements appear (“are born”) one by one; and, as formulated, these laws make sense only if there is a genuine “beginning condition” in which there are no elements at all (or at most a finite number).

The family of dynamical laws in question is described in detail in [11], where it is derived as the unique (generic) solution of certain conditions of “internal temporality”, “Bell causality” and “discrete general covariance”. For present purposes, it is enough to know that the resulting scheme describes a stochastic birth process which, “at stage  $n$ ”, yields a poset  $\tilde{C}_n$  of  $n$  elements, within which the most recently born element is maximal.

If one employs a genealogical language in which “ $x \prec y$ ” can be read as “ $x$  is an ancestor of  $y$ ”, then the  $n^{th}$  element (counting from 0) must at birth “choose” its ancestors from the elements of  $\tilde{C}_n$ , and for consistency it must choose a subset  $S$  with the property that  $x \prec y \in S \Rightarrow x \in S$ . (Every ancestor of one of my ancestors is also my ancestor.) Such a subset  $S$  (which is necessarily finite) will be called a *stem*. The dynamics is then determined fully by giving the *transition probabilities* governing each such choice of  $S \subseteq \tilde{C}_n$ . (To understand how a set of probabilities can determine a dynamics, think of a random walk. The “law of motion” of the walker is specified by giving, for each possible time and location, the probability of taking, say, a step to the right, a step to the left, or just staying put.)

We can formalize this scheme by introducing for each integer  $n = 0, 1, 2, \dots$  the set  $\tilde{\Omega}(n)$  of *labeled* causets of  $n$  elements. By definition, a member of  $\tilde{\Omega}(n)$  is thus a set  $\tilde{C}_n$  with cardinality  $n$  carrying a relation  $\prec$  such that  $x \prec y \prec z \Rightarrow x \prec z$  (transitivity), and  $x \not\prec x$  (irreflexivity), and whose elements are labeled by integers  $0, 1, \dots, n-1$  that record their order of birth. Moreover this labeling is *natural* in the sense that  $x \prec y \Rightarrow l(x) < l(y)$ ,  $l(x)$  being the label of  $x$ . Each birth of a new element occasions one of the allowed transitions from  $\tilde{\Omega}(n)$  to  $\tilde{\Omega}(n+1)$  and occurs with a specified conditional probability  $\tau$  (which turns out to depend only on a pair of simple invariants of the ancestor set  $S \subseteq \tilde{C}_n$  of the newborn).

Mathematically, however, such a set of transition probabilities  $\tau$  does not yet qualify as a stochastic process, and for good reason. In the presence of fundamental randomness, no certain predictions are possible. Instead, the “laws of motion” can at best give probabilistic answers to questions about what the object under study (causet, random walker, etc.) will do, answers of the form “Yes, with probability  $p$ ”, where  $p \in [0, 1]$ . For example, in the case of a random walk on the integers, one might ask “Will the walker ever return to the origin?”. Since the return, if it occurs at all, might be postponed to an arbitrarily late time, its probability  $p$  can be given meaning only in terms of a limiting process. For this particular question, however, there exists an obvious definition of  $p$  as the limit of an increasing sequence of probabilities, each of which can be computed from a finite number of elementary transition probabilities  $\tau$ . On the other hand, for a question like “Will the sequence of locations of the walker form an irrational decimal fraction when reduced modulo ten?”, it is not immediately clear whether any meaningful probabilistic answer can be given at all on the basis of the  $\tau$ .

In order to arrive at a definite theory, then, one needs to specify the set of questions that the theory can answer and for each one of them, explain how in principle, the “yes” probability can be computed. Fortunately, there exists a standard construction which will accomplish both these tasks starting from any consistent set of transition probabilities  $\tau$ . The result of this construction is a triad consisting of a *sample space*  $\Omega$ , a  $\sigma$ -*algebra*  $\mathcal{R}$  on  $\Omega$ , and a *probability measure*  $\mu$  with domain  $\mathcal{R}$ . In relation to the above two tasks, each member  $Q$  of  $\mathcal{R}$  corresponds to one of the answerable questions and its measure  $p = \mu(Q)$  is the answer. Such a triad constitutes what one means mathematically by a “stochastic process”, the transition probabilities  $\tau$  serve only as raw material for its construction. (That  $\mathcal{R}$  is a  $\sigma$ -algebra on  $\Omega$  means that it is a family of subsets of  $\Omega$  closed under complementation and countable intersection. That  $\mu$  is a probability measure with domain  $\mathcal{R}$  means that it takes members of  $\mathcal{R}$  to non-negative real numbers and is  $\sigma$ -additive, with  $\mu(\Omega) = 1$ . Finally,  $\sigma$ -additivity means that  $\mu$  assigns to the union of a countable collection of mutually disjoint sets in its domain the sum of the measures it assigns to the individual sets.)

In the case at hand, the sample space is the set  $\tilde{\Omega} = \tilde{\Omega}(\infty)$  of *completed labeled causets* these being the infinite causets that would result if the birth process were made to “run to completion”. (I’ll use a tilde to indicate labeling. Notice that a completed causet, though infinite, is still locally finite.\* Indeed it is past-finite in the sense that no element has more than a finite number of ancestors.) The dynamics is then given by a probability measure  $\tilde{\mu}$  constructed from the  $\tau$  whose domain  $\tilde{\mathcal{R}}$  is a  $\sigma$ -algebra which I will specify more fully in a moment. For future use, we will need in addition to  $\tilde{\Omega}$  the corresponding space  $\Omega$  of completed *unlabeled* causets, whose members can also be viewed in an obvious manner as equivalence classes within  $\tilde{\Omega}$ .

(To be pedantically precise, one should perhaps speak of the members of  $\Omega$  and  $\tilde{\Omega}$  not as single causets but as isomorphism equivalence classes of them — what one might call “abstract causets”.)

At first hearing, calling a probability measure a dynamical law might sound strange, but in fact, once we have the measure  $\tilde{\mu}$  we can say everything of a predictive nature

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\* Local finiteness, a formal realization of the concept of discreteness, is the property that the *order interval*,  $\{x | a \prec x \prec b\}$ , is finite for all elements  $a$  and  $b$ .

that it is possible to say *a priori* about the behavior of the causet  $C$ . For example, one might ask “Will the universe recollapse?” (a question analogous to our earlier question of whether the random walker would return to the origin). Mathematically, this is asking whether  $C$  will develop a “post”, defined as an element whose ancestors and descendants taken together exhaust the remainder of  $C$ . Let  $A \subseteq \tilde{\Omega}$  be the set of all completed labeled causets having posts.<sup>†</sup> Then our question is equivalent to asking whether  $C \in A$ , and the answer is “yes with probability  $\tilde{\mu}(A)$ .” It is thus  $\tilde{\mu}$  that expresses the “laws of motion” (or better “laws of growth”) that constitute our stochastic dynamics: its domain  $\tilde{\mathcal{R}}$  tells us which questions the laws can answer, and its values  $\tilde{\mu}(A)$  tell us what the answers are.

Of course, this sketch of how a sequential growth model is built up is incomplete, because I haven’t explained the construction that leads from the transition probabilities  $\tau$  to the measure  $\tilde{\mu}$ . The full details of this construction can be found in many textbooks of probability theory (e.g. [12]), but for present purposes, all we really need to know is how the domain  $\tilde{\mathcal{R}}$  of  $\tilde{\mu}$  is obtained. To each finite causet  $\tilde{S} \in \tilde{\Omega}(n)$  one can associate the so called “cylinder set” comprising all those  $\tilde{C} \in \tilde{\Omega}$  whose first  $n$  elements (those labeled  $0 \cdots n-1$ ) form an isomorphic copy of  $\tilde{S}$ ; and  $\tilde{\mathcal{R}}$  is then the smallest  $\sigma$ -algebra containing all these cylinder sets. More constructively,  $\tilde{\mathcal{R}}$  is the collection of all subsets of  $\tilde{\Omega}$  which can be built up from the cylinder sets by a countable process involving union, intersection and complementation.<sup>‡</sup>

Finally, what about the transition probabilities  $\tau$  themselves, on which the whole construction is based? Modulo certain non-generic solutions, the possibilities for the  $\tau$  have been classified in [11], the main conclusion being that  $\tau$  generically takes the form  $\tau = \lambda(\varpi, m)/\lambda(n, 0)$  where, for the potential transition in question,  $\varpi$  is the number of ancestors of the new element,  $m$  the number of its “parents”, and  $n$  the number of elements present before the birth, and where  $\lambda(\varpi, m)$  is given by the formula  $\sum_k \binom{\varpi-m}{k-m} t_k$  with the

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<sup>†</sup> One can show that  $A \in \tilde{\mathcal{R}}$ , so that  $\tilde{\mu}(A)$  is defined.

<sup>‡</sup> A slightly bigger  $\sigma$ -algebra than  $\tilde{\mathcal{R}}$  can be obtained by adjoining the sets of  $\tilde{\mu}$ -measure zero, but what these sets are will depend in general on the specific dynamical law, as determined, e.g., by a choice of the parameters  $t_n$  of [11].

$t_k$  being the free parameters or “coupling constants” of the theory. (For more details see [11] or [13].)

#### 4. Two meanings of general covariance

It might seem strange that our growth law has been expressed in terms of *labeled* causet. After all, labels in this discrete setting are the analogs of coordinates in the continuum, and the first lesson of general relativity is precisely that such arbitrary identifiers must be regarded as physically meaningless: the elements of spacetime — or of the causet — have individuality only to the extent that they acquire it from the pattern of their relations to the other elements. It is therefore natural to introduce a principle of “discrete general covariance” according to which “the labels are physically meaningless”.

But why have labels at all then? For causet, the reason is that we don’t know otherwise how to formulate the idea of sequential growth, or the condition thereon of Bell causality, which plays a crucial role in deriving the dynamics [11]. Ideally perhaps, one would formulate the theory so that labels never entered, but so far, no one knows how to do this — anymore than one knows how to formulate general relativity without introducing extra gauge degrees of freedom that then have to be canceled against the diffeomorphism invariance.

Given the dynamics as we *can* formulate it, discrete general covariance plays a double role. On one hand it serves to limit the possible choices of the *transition probabilities* in such a way that the labels drop out of certain “net probabilities”, a condition made precise in [11]. This is meant to be the analog of requiring the gravitational action-integral  $S$  to be invariant under diffeomorphisms (whence, in virtue of the further assumption of locality, it must be the integral of a local scalar concomitant of the metric). On the other hand, general covariance limits the *questions* one can meaningfully ask about the causet (cf. Einstein’s “hole argument” [14]). It is this second limitation that is related to the “problem of time”, and it is only this aspect of discrete general covariance that I am addressing in the present talk.

Just as in the continuum the demand of diffeomorphism-invariance makes it harder to formulate meaningful statements,\* so also for causet the demand of discrete general

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\* Think, for example, of the statement that light slows down when passing near the sun.



covariance has the same consequence, bringing with it the risk that, even if we succeed in characterizing the covariant questions in abstract formal terms, we may never know what they mean in a physically useful way. I believe that a similar issue will arise in every approach to quantum gravity, discrete or continuous (unless of course general covariance is renounced).<sup>†</sup> However, it seems fair to say that both the nature of the difficulty and the manner of its proposed resolution will appear with special clarity in the context of causal sets, whose discreteness removes many of the technical difficulties that tend to obscure the underlying physical issues in the continuum.

## 5. What are the covariant questions?

Given the formal developments of Section 4, it is not hard to see which members of  $\tilde{\mathcal{R}}$  express covariant predicates, and from a dynamical point of view, these are the only covariant predicates of interest.

Before describing them, let me illustrate the issue we face with the question, “Which (unlabeled) causet is formed by the first  $n$  elements to be born?”. In effect, we are asking for the probability distribution induced by our measure  $\tilde{\mu}$  on the space  $\Omega(n)$  of unlabeled  $n$ -orders, but, although such a distribution can be computed, it has no obvious meaning because a question like “Do the first three elements of  $C$  form a 3-chain<sup>b</sup>?” has in general different answers depending on what order of birth you impute to the elements of  $C$ . Thus if we were to divide the set  $\tilde{\Omega}$  into two parts, the “yes” part composed of those  $\tilde{C} \in \tilde{\Omega}$  whose first 3 elements make up a chain, and the “no” part composed of those  $\tilde{C}$  whose first 3 elements make up one of the other four 3-orders, then some members of the “yes” set would be isomorphic to members of the “no” set.

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<sup>†</sup> In the case of canonical quantum gravity, this issue *is* the problem of time. There, covariance means commuting with the constraints, and the problem is how to interpret quantities which do so in any recognizable spacetime language. For an attempt in string theory to grapple with similar issues see [15].

<sup>b</sup> A (finite) chain is a causet whose elements can be arranged so that each is an ancestor of the next. For a 3-chain, we have three elements  $a$ ,  $b$ , and  $c$  such that  $a \prec b \prec c$ .

We see now what it means for a subset of  $A \subseteq \tilde{\Omega}$  to be covariant: it cannot contain any labeled completed causet  $\tilde{C}$  without containing at the same time all those  $\tilde{C}'$  isomorphic to  $\tilde{C}$  (i.e. differing only in their labelings). To be measurable as well as covariant,  $A$  must also belong to  $\tilde{\mathcal{R}}$ . Let  $\mathcal{R}$  be the collection of all such sets:  $A \in \mathcal{R} \iff A \in \tilde{\mathcal{R}}$  and  $\forall \tilde{C}_1 \simeq \tilde{C}_2 \in \tilde{\Omega}, \tilde{C}_1 \in A \Rightarrow \tilde{C}_2 \in A$ . It is not hard to see that  $\mathcal{R}$  is a sub- $\sigma$ -algebra of  $\tilde{\mathcal{R}}$ , whence the restriction of  $\tilde{\mu}$  to  $\mathcal{R}$  is a measure  $\mu$  on the space  $\Omega$  of unlabeled completed causets.\* It is this measure  $\mu$  that provides the answers to all the covariant questions for which the dynamics has answers.† But what do these questions signify physically?

## 6. Stem sets and a conjecture

Among the questions belonging to  $\mathcal{R}$  there are some which do have a clear significance. Let  $S \in \Omega(n)$  be any finite unlabeled causet and let  $R(S) \subseteq \Omega$  be the “stem set”,  $\{C \in \Omega | C \text{ admits } S \text{ as a stem}\}$ . (Thus  $R(S)$  comprises those unlabeled completed causets with the property that, with respect to some natural labeling, the first  $n$  elements form a causet isomorphic to  $S$ .) Since (as one can prove)  $R(S)$  is measurable, it belongs to  $\mathcal{R}$ . For this particular element of  $\mathcal{R}$ , the meaning of the corresponding causet question is evident: “Does the causet possess  $S$  as a stem?”.<sup>b</sup> Equally evident is the significance of any question built up as a logical combination of stem-questions of this sort. To such compound stem-questions belong members of  $\mathcal{R}$  built up from stem-sets  $R(S)$  using union, intersection and complementation (corresponding to the logical operators ‘or’, ‘and’ and ‘not’). If all the members of  $\mathcal{R}$  were of this type, we would not only have succeeded in

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\* As just defined, an element  $A \in \mathcal{R}$  is a subset of  $\tilde{\Omega}$ . However, because it is re-labeling invariant, it can also be regarded as a subset of  $\Omega$ , an equivalence which I will henceforth utilize without explicit mention.

† Notice the distinction that arises here between a subset of  $\tilde{\Omega}$  that fails to belong to  $\mathcal{R}$  and one that fails even to be covariant (one that cannot be regarded as a subset of  $\Omega$ ). The former corresponds to a question that the dynamics can’t answer, the latter to a question that is without any physical meaning at all.

<sup>b</sup> Here is a simple example. Let  $\Lambda$  be the 3-element causet given by  $a \prec c, b \prec c$ , let  $V$  be its dual (given by  $a \succ c, b \succ c$ ), and let  $S$  be the 2-chain (given by  $a \prec b$ ). Then  $V$  possesses  $S$  as a stem, while  $\Lambda$  does not.

characterizing the dynamically meaningful covariant questions at a formal level, but we would have understood their physical significance as well. <sup>\*</sup> The following conjecture asserts that, to all intents and purposes, this is the case.

**Conjecture** The “stem-sets”  $R(S)$  generate<sup>†</sup> the  $\sigma$ -algebra  $\mathcal{R}$  up to sets of measure zero.

Here the technical qualification about sets of measure zero complicates the statement of the conjecture but does not essentially weaken it. Some such qualification is required, unfortunately, because one can exhibit counterexamples to the unqualified conjecture. Notice, once again, that the words “measure zero” have meaning only with respect to some choice of stochastic dynamical law, since different choices correspond to different measures  $\mu$  which, in general will possess different families of measure-zero sets. Thus, a more complete phrasing of the conjecture would read “For every choice of sequential growth model, the stem-sets  $R(S)$  generate  $\mathcal{R}$  up to sets of measure zero.” Here a sequential growth model is any member of the “generic” family described briefly in Section 3 above (and at length in [11]), and more generally any solution of the conditions of Bell causality, etc. delineated in [11].

The conjecture asserts that  $\mathcal{R}_S = \mathcal{R}$ , where  $\mathcal{R}_S$  is the subalgebra of  $\mathcal{R}$  generated by the stem-sets  $R(S)$ . In a moment, I’ll present some evidence in favor of this equality, but first I’d like to stress that, even if  $\mathcal{R}_S$  fails to exhaust  $\mathcal{R}$ , it still supplies us with a large store of predicates whose physical significance is transparent, and this store probably suffices for practical purposes since one’s experience so far indicates that all predicates of interest belong to it, either outright or up to a set of measure zero. For example, the predicate “contains a post” belongs to  $\mathcal{R}_S$ .

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<sup>\*</sup> This is not yet their *phenomenological* meaning, of course. To get the phenomenological significance (in cases where there is one) one must translate between the combinatorial language proper to the causet and the geometrical and field-theoretic language of macroscopic physics.

<sup>†</sup> A family  $\mathcal{F}$  of subsets is said to *generate* a  $\sigma$ -algebra  $\mathcal{A}$  if  $\mathcal{A}$  is the smallest  $\sigma$ -algebra containing all the members of  $\mathcal{F}$ . For example, the cylinder sets introduced above generate the  $\sigma$ -algebra  $\tilde{\mathcal{R}}$ .

Now the intuitive import of our conjecture is that everything we need to say about a causet can be phrased in terms of its stems. If this is so, then we'd expect at least that a specification of the full set of stems of a given causet  $C \in \Omega$  would suffice to characterize  $C$  uniquely. As a matter of fact, not all causets enjoy this feature, but the exceptions are rare enough that the probability of one of them being produced by a sequential growth process is zero. (Employing the jargon of probability theory, we may say that a causet  $C$  produced by one of the sequential growth processes is *almost surely* characterized by its stems.) That this is so speaks in favor of the conjecture but, as far as I know, is not enough to demonstrate it fully. Its proof, in any case, can be given in three steps as follows [16].

- (a) If  $C \in \Omega$  *fails* to be characterized by its stems then it must contain an infinite number of copies of some stem  $S$ .
- (b) If  $C \in \Omega$  contains an infinite number of copies of the same stem  $S$  then it must contain an infinite antichain (indeed an infinite level).
- (c) The probability is zero that a  $C$  produced by one of the classical sequential growth models will contain an infinite level, or indeed any infinite antichain.

Strictly speaking, statement (c) has been proven only for the models described above which are parameterized by the “coupling constants”  $t_0, t_1, t_2, \dots$ , but these are generic in the sense that they include all solutions for which no transition probability  $\tau$  vanishes [11]. Moreover (c) fails for the special case where  $t_n = 0$  for all  $n \geq 2$ . But this exception does not affect the main conclusion, because if  $t_0$  is the only nonzero  $t_n$  then  $C$  is almost surely an infinite antichain, while if  $t_0$  and  $t_1$  are both nonzero then  $C$  is almost surely an infinite number of copies of the tree in which each element has an infinite number of children; and both these causets are also characterized by their stems.

To get a feel for what these results mean, consider the causet  $C'$  consisting of an infinite number of unrelated copies of the infinite chain  $e_0 \prec e_1 \prec e_2 \prec \dots$ . This causet is *not* characterized by its stems, because, e.g., the causet  $C''$  made from  $C'$  by adjoining a single  $n$ -chain ( $1 \leq n < \infty$ ) has precisely the same stems as  $C'$ . (A causet  $S$  is such a stem iff it is the sum of a finite number of finite chains.) The proof outlined above then tells us that  $C'$  and  $C''$  are infinitely unlikely to “grow” in any of the models with  $t_2 > 0$ .

## 7. Conclusion and some further questions

General covariance in classical gravity is a two-edged sword. On one hand it is both philosophically satisfying (at least to those of us who favor “relational” or “dialectical” theories) and heuristically fruitful for the way in which it limits the possible equations of motion. On the other hand it forces the consequent formalism farther from experience, because it renders meaningless all statements which are not, at least implicitly, of a global character. (In this sense, the context for a generally covariant theory is always cosmological in scope.) The meaningful statements become thereby both harder to formulate and harder to interpret (“problem of time”).

In the case of (classical) general relativity, these difficulties are somewhat mitigated by the theory’s determinism, which means that predictions can effectively be made and verified locally. Such is not the case, however, in an indeterministic theory, where the dynamics must presumably be expressed via relations of a probabilistic character among covariant statements whose global character is much harder to circumvent. The problem of characterizing and interpreting such statements must therefore arise in virtually every approach to quantum gravity, including of course the causet approach.<sup>b</sup>

But the classical sequential growth models of causet dynamics are also indeterministic, and consequently the same problems arise for them. These models, which are intended primarily as a (practical and conceptual) stepping stone to the quantum theory of causets, offer a precisely defined schema within which the interpretational difficulties deriving from general covariance can be confronted. And within this setting, we have largely resolved them: the dynamically meaningful predicates have been precisely characterized in Section 5, and an interpretation for many or all of them (in terms of “stem-predicates”) has been set forth in Section 6.

For an ordinary stochastic process unfolding against the background of some non-dynamical parameter time, the predicates corresponding to measurable sets of trajectories can all be built up as logical combinations of more elementary ones that acquire their full meaning in a finite time. (They don’t refer to the infinite future). So it is also for the

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<sup>b</sup> Among the possible exceptions would be approaches which attempt to restore strict determinism.

“cylinder set” predicates introduced in Section 3, but these predicates are unfortunately meaningless *per se* since they are not generally covariant. The nearest covariant replacements for these cylinder sets would seem to be the stem-sets  $R(S)$ . Unlike the cylinder sets, they do refer to the infinite future, for, although the predicate corresponding to a stem-set can become true in a finite time (after the birth of a finite number  $n$  of elements) it can only strictly speaking become false in the limit  $n \rightarrow \infty$ .<sup>\*</sup> Nonetheless, falsehood can become certain “for all practical purposes”, and in this sense, the truth or falsehood of a stem-predicate is also “verifiable in a finite time”.

Moreover the physical significance of a stem-predicate is transparent, since its truth is decided simply by whether the “universe” (i.e. the actual causet) does or does not (never will) contain the stem in question. To the extent, then, that all assertions of dynamical interest can be built up from the stem predicates, we will have resolved the “problem of time” for this range of models. A proof of the conjecture of Section 6 would vouchsafe us this conclusion. It is clear, moreover, that this relatively satisfactory situation depends heavily on the discreteness of the causal set.

To what extent can we hope to repeat these same steps in the quantum case (the case of quantum gravity)? As a mathematical object, a quantum measure/decoherence functional is not that different from a classical measure like  $\mu$  above, so one might hope much of the above would go through relatively unchanged. Two hurdles arise immediately, however. First, of course, the predictive meaning of a quantum measure is much less well understood than that of a classical probability measure, whence, even if we could say precisely which subsets of  $\Omega$  were “measurable”, we still would not know exactly how to use this information.<sup>†</sup> Second, the theorems that above led us from the transition probabilities  $\tau$  to the measure  $\tilde{\mu}$  (and thence to  $\mu$ ) break down in the quantum case because the complex amplitudes that one is constructing are no longer necessarily bounded in absolute value, unlike probabilities which are confined to the compact space  $[0, 1] \subseteq \mathbb{R}$ . How serious these

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<sup>\*</sup> Even if a stem  $S$  has not appeared yet, we can never be absolutely certain that it won’t appear later on.

<sup>†</sup> For an attempt to codify the predictive use of the quantum measure without first reducing it to a set of classical probabilities, see [17].

problems are is hard to say, but certainly the second of them means that further technical developments would need to occur before one could rigorously state a quantum analog of the conjecture of Section 6.

Finally, I want to raise — without attempting to settle it — a question about the status of conditional probability in quantum gravity, or rather in the classical analog of causal quantum gravity which has been the basis of our considerations here. Certainly, one can “relativize” the measure  $\mu$  to any measurable subset of  $\Omega$  and thereby lend meaning to questions of the sort “Given that  $S_1$  occurs as a stem in  $C$  (the universe), what is the probability that  $S_2$  also occurs as a stem?”. If such questions are the only ones we need to consider, then conditional probabilities here will have the same derived status as they do in other branches of probability theory. But how clear is it that such a syntax captures what we’d really like to ask? Consider instead something like this: “Given that  $S_1$  occurs as a stem in  $C$  what is the probability that a second stem  $S_2$  occurs containing *this particular copy of  $S_1$*  (the  $S_1$  that “we inhabit”)?”. When  $C$  contains more than one stem isomorphic to  $S_1$ , this second type of question seems different from the first, and indeed not even clearly defined on the sole basis of the “absolute” measure  $\mu$ . Does this mean we need a *logically independent* concept of conditional probability and an extended formalism to express it, which would re-open the “problem of time” in a new context? Or is it enough to remark that stems of sufficient complexity are unlikely to occur more than once, whence the problem is absent in practice? As with other conceptual issues raised by quantum gravity, so also with this “this particular stem” issue, it’s hard to say whether its resolution demands deep thought or just a bit of progress on the technical front.

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